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On Kelvin–Helmholtz instability in a rotating fluid

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Chandrasekhar's (1961) solution to the eigenvalue equation arising from the Kelvin–Helmholtz stability problem for a rotating fluid is shown to be incorrect. The unstable modes are correctly enumerated with the aid of Cauchy's principle of the argument. Various previously published solutions using Chandrasekhar's analysis are corrected and extended.

1. Introduction

Chandrasekhar (1961, §105) devotes a section of his treatise to the investigation of the effect of rotation on the development of Kelvin–Helmholtz instability. Two uniform, horizontally-superposed fluids of different densities are in relative horizontal motion and are rotating with uniform angular velocity about a vertical axis. Perturbing the solution with a travelling wave in the direction of streaming, imposing the boundary conditions at the interface, and linearizing, Chandrasekhar obtains the eigenvalue equation for the wave speed. He then uses a graphical method, together with the fundamental theorem of algebra, to enumerate the eigenvalues. His argument necessitates determining the singular points of the eigenvalue equation. We show that Chandrasekhar's determination of these singular points is incomplete, leading to erroneous results. With the aid of Cauchy's principle of the argument, we enumerate the eigenvalues and present a simple, sufficient condition for stability.

Alterman, in a series of papers (1961 *a*, *b*, *c*) dealing with the Kelvin–Helmholtz stability problem under various force fields, uses Chandrasekhar's results and generalizations of his method, in consequence of which her results are incorrect. We present the correct results and, in one instance, extend our analysis to solve the problem for a more general flow configuration than that treated by Alterman.

2. Chandrasekhar's analysis

Chandrasekhar considers two uniform, superposed fluids of densities ρ_1, ρ_2 having velocities U_1, U_2 in the x direction and in a state of uniform rotation about the z-axis with an angular velocity Ω. [In what follows a subscript 1 (2) refers to the lower (upper) fluid; ρ_1 is greater than ρ_2.] Imposing upon the steady-state solution a small disturbance whose dependence on x and t is given by

$$\exp[ik(x-ct)],$$

where k is the wave number and c the wave speed, linearizing the equations of motion, and applying the conditions of continuity of pressure and normal velocity at the undisturbed interface, Chandrasekhar obtains the eigenvalue equation

$$\alpha_1(U_1-c)^2[1-4\Omega^2k^{-2}(U_1-c)^{-2}]^{\frac{1}{2}}+\alpha_2(U_2-c)^2[1-4\Omega^2k^{-2}(U_2-c)^{-2}]^{\frac{1}{2}} -gk^{-1}(\alpha_1-\alpha_2)-kT(\rho_1+\rho_2)^{-1}=0, \quad (2.1)$$

where $$\alpha_i=\rho_i/(\rho_1+\rho_2) \quad (i=1,2), \quad (2.2a,b)$$

g is the acceleration due to gravity, and T is the surface tension. The square roots of (2.1) must be taken to have positive real parts since the product of wave number and the inverse of these real parts represents the rate of decay of the disturbance with increasing distance from the interface.

Applying the transformation

$$U_1-c=\xi(g/k)^{\frac{1}{2}}, \quad U_2-c=\eta(g/k)^{\frac{1}{2}}, \quad (2.3a,b)$$

originally due to Taylor (1931), and neglecting surface tension, Chandrasekhar reduces the problem to the simultaneous equations

$$\alpha_1\xi^2(1-\omega^2\xi^{-2})^{\frac{1}{2}}+\alpha_2\eta^2(1-\omega^2\eta^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2 \quad (2.4)$$

and $$\xi-\eta=\tilde{V}, \quad (2.5)$$

where $$\omega^2=4\Omega^2/(gk), \quad \tilde{V}=(U_1-U_2)(k/g)^{\frac{1}{2}}. \quad (2.6a,b)$$

In order to apply the fundamental theorm of algebra, which is not directly applicable to (2.4), Chandrasekhar introduces the following set of four equations:

$$\alpha_1\xi^2(1-\omega^2\xi^{-2})^{\frac{1}{2}}+\alpha_2\eta^2(1-\omega^2\eta^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2, \quad (2.7)$$

$$\alpha_1\xi^2(1-\omega^2\xi^{-2})^{\frac{1}{2}}-\alpha_2\eta^2(1-\omega^2\eta^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2, \quad (2.8)$$

$$-\alpha_1\xi^2(1-\omega^2\xi^{-2})^{\frac{1}{2}}+\alpha_2\eta^2(1-\omega^2\eta^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2, \quad (2.9)$$

$$-\alpha_1\xi^2(1-\omega^2\xi^{-2})^{\frac{1}{2}}-\alpha_2\eta^2(1-\omega^2\eta^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2. \quad (2.10)$$

He then states that, from the fundamental theorem of algebra, equations (2.7) to (2.10) 'have a total of exactly eight roots ... [which] are continuous functions of the parameters of the equation[s] except at the singular points $(\pm\xi_0, \pm\omega)$ and $(\pm\omega, \pm\eta_0)$, ... where ξ_0 and η_0 are determined by the equations

$$\alpha_1\xi^2(1-\omega^2\xi^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2 \quad \text{and} \quad \alpha_2\eta^2(1-\omega^2\eta^{-2})^{\frac{1}{2}}=\alpha_1-\alpha_2\text{'}. \quad (2.11)$$

Using these two facts, Chandrasekhar determines the number of roots of each of (2.7)–(2.10) as follows.

The equations are solved for the particular case $\tilde{V}=0$ $(\xi=\eta)$. $|\tilde{V}|$ is then increased until the first singular point is reached, $|\tilde{V}|=\xi_0-\omega\equiv V_1$, say. At this point the equations may exchange roots. Using the enunciated theorem and the graphs of the equations, Chandrasekhar determines the number of roots of each equation at $|\tilde{V}|=V_1+$. This procedure is then continued, Chandrasekhar considering the exchange of roots at each of the singular points quoted above. However, $(\pm\xi_0, 0)$ and $(0, \pm\eta_0)$ are also singular points, in consequence of which Chandrasekhar's results are incorrect.† Furthermore, the inclusion of these

† Prof. Chandrasekhar informs me that he is aware of this error and will present an amended treatment in a second printing of his book.

singular points renders the method inapplicable, as it is not possible, by Chandrasekhar's method, to determine the exchange of roots between (2.7) and(2.8) at $(\pm\xi_0, 0)$ and between (2.7) and (2.9) at $(0, \pm\eta_0)$.

In the next section we use Cauchy's principle of the argument to yield the correct enumeration of the eigenvalues of (2.1).

3. Cauchy's principle of the argument

Introducing the transformations

$$c = \tfrac{1}{2}(U_1+U_2)+Vs, \quad V = \tfrac{1}{2}(U_1-U_2), \qquad (3.1\,a,b)$$

$$\alpha_2 = \sigma\alpha_1, \quad \kappa = kV^2/g, \quad x = 2\Omega/(kV), \quad \tau = k^2T/(\rho_1 g), \qquad (3.2\,a,b,c,d)$$

we write the eigenvalue equation as

$$F(s) \equiv \sigma-\tau-1+\sigma\kappa(s+1)^2[1-x^2(s+1)^{-2}]^{\frac{1}{2}}+\kappa(s-1)^2[1-x^2(s-1)^{-2}]^{\frac{1}{2}} = 0, \quad (3.3)$$

where the square roots are to have positive real parts. (From here on we assume V, and hence x, to be positive; the results are not altered if V is negative.)

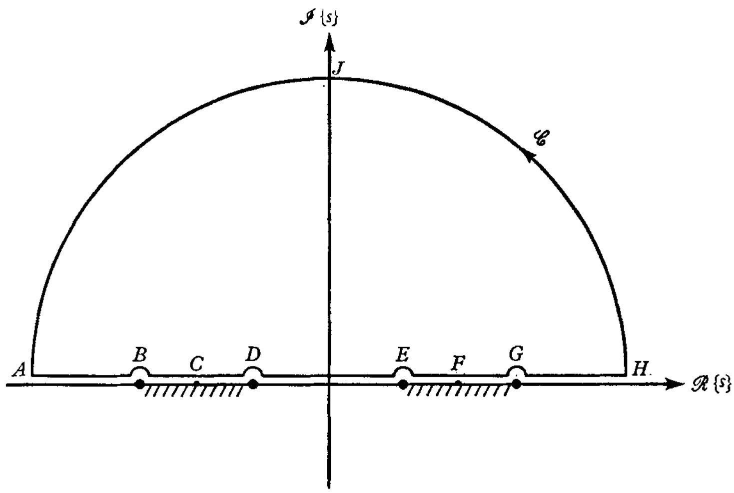

FIGURE 1. The contour in the s-plane to which Cauchy's principle of the argument is applied. D may be to the right of E and the branch cuts overlap.

To render $F(s)$ analytic, we introduce branch cuts in the s-plane extending along the real axis from $-1-x$ to $-1+x$ and from $1-x$ to $1+x$.

In order to determine the solutions of (3.3) we apply Cauchy's principle of the argument: for any function $f(z)$, analytic within and on a closed contour $\mathscr{C}$, the number of zeros minus the number of poles of $f(z)$ within $\mathscr{C}$ is $(1/2\pi)$ times the increase of $\arg f(z)$ as z traverses $\mathscr{C}$ once in an anticlockwise direction (see Copson 1962, §6.2). We apply this principle to $F(s)$ and take $\mathscr{C}$ to be the real axis of the s-plane, indented above the branch cuts, and a semi-circle whose radius tends to infinity in the upper half plane. A typical contour is shown in figure 1.

To construct the map of $\mathscr{C}$ onto the F-plane (the Cauchy–Nyquist diagram), we define $x_1(\sigma)$ as that value of x for which

$$\kappa^{-1}F'(s) = \sigma[2(s+1)^2-x^2]\{(s+1)[1-x^2(s+1)^{-2}]^{\frac{1}{2}}\}^{-1} + [2(s-1)^2-x^2]\{(s-1)[1-x^2(s-1)^{-2}]^{\frac{1}{2}}\}^{-1} = 0 \quad (3.4)$$

has a double root in $-1+x < s < 1-x$ (or, equivalently, that value of x for which $F''(s) = 0$ in this range of s). $x_1(\sigma)$, which is determined numerically, is shown in figure 2. For s in the range $(-1+x, 1-x)$, $F(s)$ has one maximum if

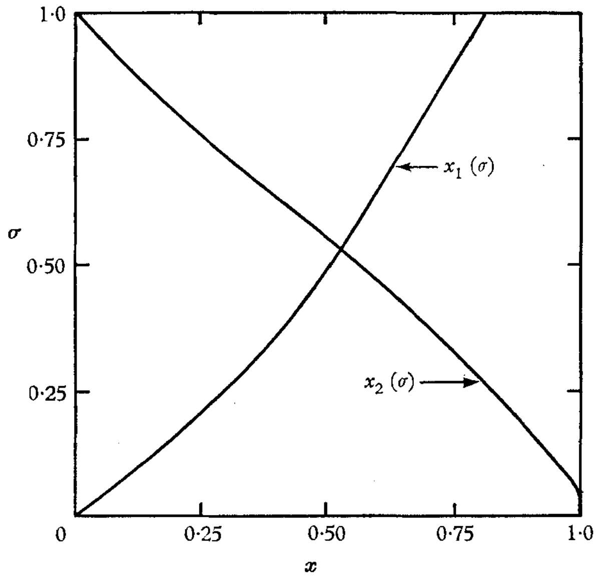

FIGURE 2. The curves $x_1(\sigma)$, $x_2(\sigma)$.

$x \geqslant x_1(\sigma)$ and two maxima and one minimum if $x < x_1(\sigma)$. We label the values of s, at the maxima a and b, $F(a) \equiv F_a \geqslant F(b) \equiv F_b$, and at the minimum m, $F(m) \equiv F_m$.

We also define $x_2(\sigma)$, determined numerically and shown in figure 2, by

$$F_a = F(1) \equiv F_1 \quad \text{for} \quad x = x_2(\sigma).$$

It can be shown that $x \lesseqgtr x_2(\sigma)$ implies $F_a \gtreqless F_1$. In addition, the stipulation that the square roots of (3.3) have positive real parts requires that for all x

$$[1-x^2(s+1)^{-2}]^{\frac{1}{2}} = i[x^2(s+1)^{-2}-1]^{\frac{1}{2}}\,\mathrm{sgn}\,(s+1) \quad (s \text{ on } BD), \tag{3.5}$$

$$[1-x^2(s-1)^{-2}]^{\frac{1}{2}} = i[x^2(s-1)^{-2}-1]^{\frac{1}{2}}\,\mathrm{sgn}\,(s-1) \quad (s \text{ on } EG). \tag{3.6}$$

The construction of the Cauchy–Nyquist diagrams, examples of which are shown in figure 3, is now straightforward.

To enumerate the complex eigenvalues, we determine the number of times the Cauchy–Nyquist diagram encircles the origin [from Cauchy's principle and the absence of poles of $F(s)$ within $\mathscr{C}$, this is the number of pairs of complex conjugate eigenvalues]. The position of the origin in figure 3 is dependent upon the particular values of x, σ, κ, τ. Exploring all possibilities (there are approximately thirty), we determine the number of complex eigenvalues as shown in table 1 [$F_{-1} \equiv F(-1)$ therein]. From the table, we see that: there cannot be

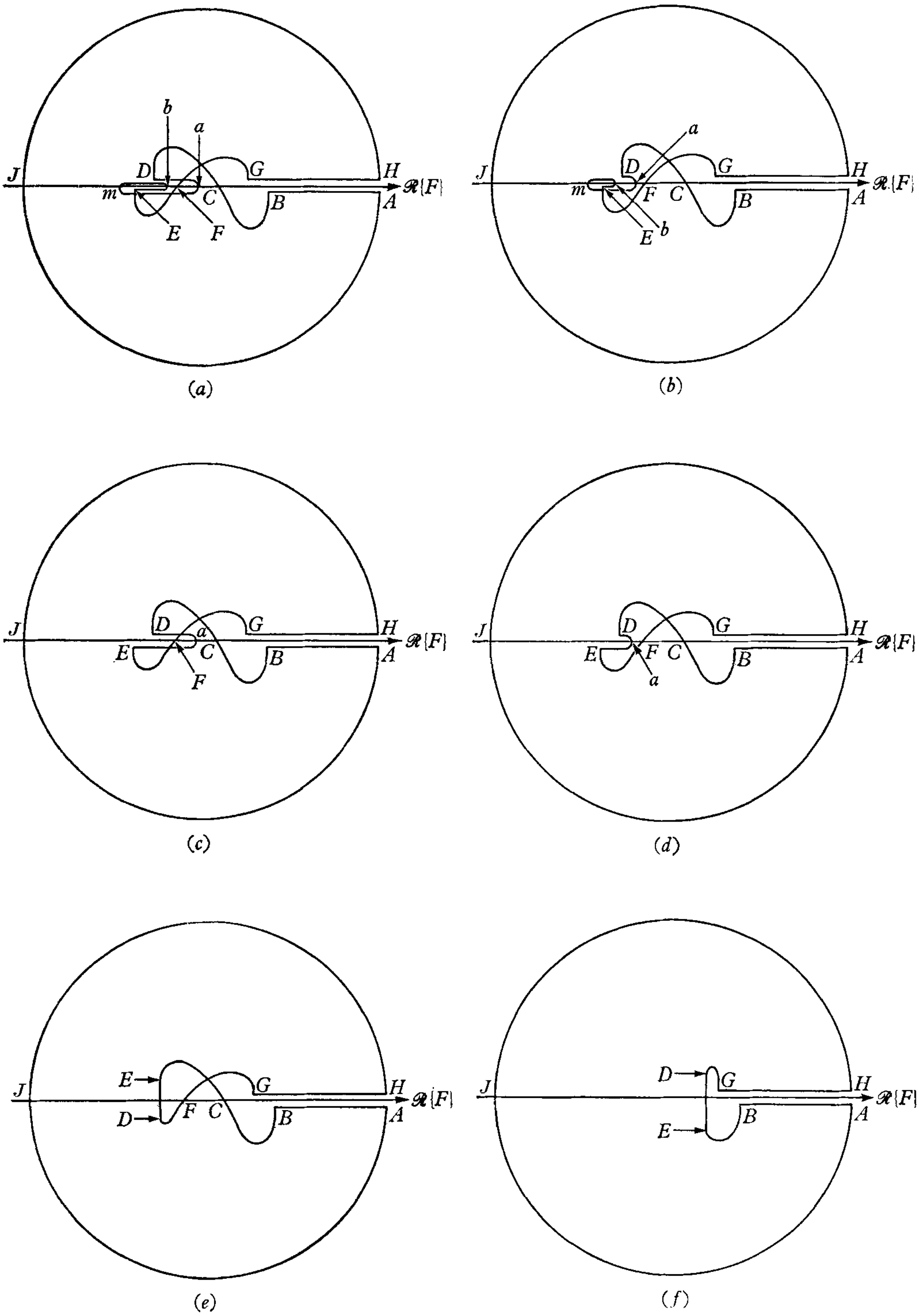

FIGURE 3. The Cauchy–Nyquist diagrams. (*a*) $x < x_1, x_2$. (*b*) $x_2 < x < x_1$. (*c*) $x_1 < x < x_2$. (*d*) $x_1, x_2 < x < 1$. (*e*) $1 < x < 2$. (*f*) $x > 2$.

instability for $x \geqslant 2$; a sufficient condition for stability† in $x < 2$ is

$$1-\sigma+\tau \geqslant 4\kappa[1-(x^2/4)]^{\frac{1}{2}} \quad (x < 2). \tag{3.7}‡$$

	x	F	N
(i)	$[2, \infty)$		0
(ii)	$(0, 2)$	$F_{-1} \leqslant 0$	0
(iii)	$(0, x_2)$	$F_a \geqslant 0, F_1 \leqslant 0$	0
(iv)	$(0, x_1)$	$F_b \geqslant 0, F_m \leqslant 0$	0
(v)	$[1, 2)$	$F_1 > 0$	2
(vi)	$(x_2, 1)$	$F_1 > 0, F_a < 0$	2
(vii)	Otherwise		1

TABLE 1. The distribution of complex eigenvalues. N is the number of pairs of complex conjugate eigenvalues in the indicated range of x

For max $(x_1, x_2) \leqslant x \leqslant 2$ this is also a necessary condition. For $x <$ max (x_1, x_2), we see from lines (iii) and (iv) of table 1 that there may be a region of stability for which (3.7) is not satisfied. Returning to physical variables, we find that a sufficient condition for stability is

$$(U_1 - U_2)^2 \leqslant 2\Omega^2 k^{-2} + k^{-2}\{4\Omega^4 + k^2\rho_1^{-2}[g(\rho_1-\rho_2) + k^2 T]^2\}^{\frac{1}{2}}. \tag{3.8}$$

A table enumerating the real eigenvalues can be easily obtained, but is of excessive length and hence is omitted. It can be simply seen from the Cauchy–Nyquist diagrams, however, that the number of real eigenvalues varies between zero and four. We can also show that the total number of eigenvalues is never more than four. Finally, we note from the Cauchy–Nyquist diagrams that if $F_{-1} < 0 \leqslant F(1+x)$ (3.3) has no solution whatever, in which case the original linearized perturbation equation does not admit a discrete spectrum solution.

Generalizing Chandrasekhar's argument to include the effect of surface tension, Alterman (1961 *a*) asserts that the system is unstable to long waves. From (3.8) we see that the actual condition for stability is

$$(U_1 - U_2)^2 \leqslant 4\Omega^2 k^{-2} + O(1) \quad (k \to 0), \tag{3.9}$$

and hence there is stability to a long wave length disturbance.

In a later publication, Alterman (1961 *b*) considers the problem of two heterogeneous fluids, with horizontal velocities U_1, U_2 in the same direction, separated by a horizontal interface at $z = 0$, the densities being given by

$$\rho = \rho_1 e^{-\beta z} \quad (z < 0), \quad \rho = \rho_2 e^{-\beta z} \quad (z > 0). \tag{3.10 a, b}$$

Invoking the Boussinesq approximation, she shows that the eigenvalue equation governing stability is formally equivalent to (2.1) if Ω^2 is replaced by $\frac{1}{4}\beta g$. The correct sufficient condition for stability is hence given by (3.8) once this replace-

† By stability here we mean stability to an *exponentially* growing disturbance; the assumed form of the disturbance rules out any possibility of investigating algebraic instability, for which an initial value approach is required.

‡ In Chandrasekhar's notation, this becomes $|\tilde{V}| \leqslant \xi_0$ after setting $\tau = 0$.

ment has been made. Generalizing Alterman's model such that the fluid densities are given by

$$\rho = \rho_1 e^{-\beta_1 z} \quad (z < 0), \quad \rho = \rho_2 e^{-\beta_2 z} \quad (z > 0), \qquad (3.11\,a, b)$$

we obtain the eigenvalue equation

$$\alpha_1(U_1-c)^2[1-\beta_1 g k^{-2}(U_1-c)^{-2}]^{\frac{1}{2}}+\alpha_2(U_2-c)^2[1-\beta_2 g k^{-2}(U_2-c)^{-2}]^{\frac{1}{2}} \\ -gk^{-1}(\alpha_1-\alpha_2)-kT(\rho_1+\rho_2)^{-1}=0, \qquad (3.12)$$

and the eigenvalues can be enumerated in the same manner as before. For the sake of brevity it suffices to say that (3.8) is a sufficient condition for stability if Ω^2 is replaced by $(g/4)\min(\beta_1, \beta_2)$.

In another paper, Alterman (1961 *c*) obtains the eigenvalue equation pertinent to the fluid system originally considered by Chandrasekhar, with the added condition that the fluid be a perfect conductor under the influence of a uniform, horizontal magnetic field of intensity H. In the limit as the wave length of the disturbance tends to infinity, Alterman obtains the equation (3.12) with $\beta_i g$ replaced by $4\Omega^2+(\mu H^2k^2/2\pi\rho_i)\,(i=1,2)$. Her sufficient condition for stability should be replaced by

$$(U_1-U_2)^2 \leqslant 2\Omega^2k^{-2}+(\mu H^2/4\pi\rho_1)+k^{-2}\{4[\Omega^2+(\mu H^2k^2/8\pi\rho_1)]^2+k^2\rho_1^{-2}[g(\rho_1-\rho_2) \\ +k^2T]^2\}^{\frac{1}{2}}. \qquad (3.13)$$

4. Conclusion

We conclude that an eigenvalue equation of the form (2.1) arises in various Kelvin–Helmholtz stability problems and that applying Cauchy's principle of the argument is a simple and efficient method of enumerating the eigenvalues.

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